Organization of modular networks

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We examine the global organization of heterogeneous equilibrium networks consisting of a number of well-distinguished interconnected parts—"communities" or modules. We develop an analytical approach allowing us to obtain the statistics of connected components and the intervertex distance distribution in these modular networks, and to describe their global organization and structure. In particular, we study the evolution of the intervertex distance distribution with an increasing number of interlinks connecting two infinitely large uncorrelated networks. We demonstrate that even a relatively small number of shortcuts unite the networks into one. In more precise terms, if the number of interlinks is any finite fraction of the total number of connections, then the intervertex distance distribution approaches a δ-function peaked form, and so the network is united.

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I. INTRODUCTION

Many real-world networks contain distinct parts with different architectures. In this sense, they are strongly heterogeneous. For example, the Internet—the net of physically interconnected computers—is connected to mobile cellular networks. One should note that the issue of network heterogeneity is among the key problems in the statistical mechanics of complex networks [1–7]. The question is how does the network's inhomogeneity influence its global structure? The quantitative description of the global organization of a network is essentially based on the statistics of the nth component of a vertex in the network, in particular, on the size statistics [8–10]. The nth component of a vertex is defined as a set of vertices which are not farther than distance n from a given vertex. From this statistics, one can easily find less informative but very useful characteristics—the distribution of intervertex distances and its first moment, the average intervertex distance. In networks with the small-world phenomenon, so-called small worlds, the mean length of the shortest path $\bar{\ell}(N)$ between two vertices grows more slowly than any positive power of the network size N (the total number of vertices). Rather typically, $\bar{\ell}(N) \sim \ln N$. As a rule, in infinite small worlds, the distribution of intervertex distances approaches a δ-function form, where the mean width $\delta\ell$ is much smaller than $\bar{\ell}$. Moreover, in uncorrelated networks, $\delta\ell(N \to \infty) \to$ const. So, in simple terms, vertices in these infinite networks are almost surely mutually equidistant. This statement can be easily understood if a network has no weakly connected separate parts [11]. In this paper we consider a contrasting situation. Our networks are divided into a number of nonoverlapping but interlinked subnetworks, say $j = 1, 2, \ldots, m$. It is important that we suppose the connections between these subnetworks to be organized dif-

ferently from those inside them (see Fig. 1). This assumption results in a global (or one may say, macroscopic) heterogeneity of the network. Using the popular term "community," one can say that our networks have well-distinguished communities or modules. Modular architectures of this kind lead to a variety of effects [12–15]. Figure 1 explains the difference between these modular networks and the well studied m-partite networks [16,17].

In this work we analytically describe the statistics of the nth components in these networks when all m communities are uncorrelated. For the sake of brevity, here we consider only the case of $m=2$, i.e., of two networks with shortcuts between them. As an immediate application of this theory we
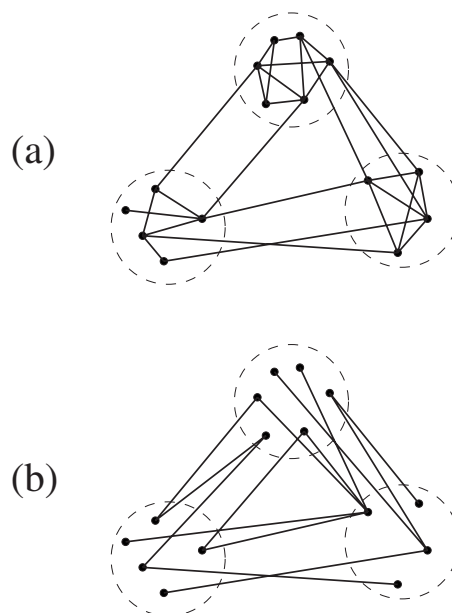

FIG. 1. (a) Example of the network that we study in this paper in the case $m=3$. The structure of interconnections between the three nonoverlapping subnetworks differs from the structure of connections inside these subnetworks. Moreover, the structures of the three subnetworks may differ. (b) A contrasting example of a tripartite graph, where connections between vertices of the same kind are absent.

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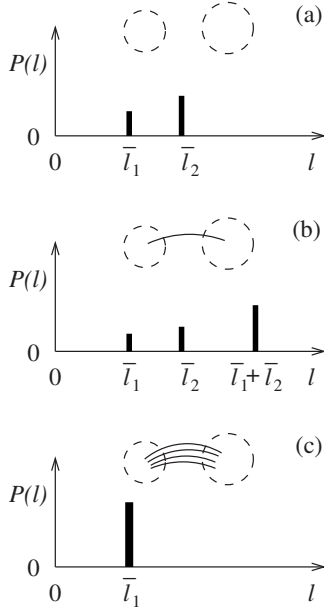‡samukhin@ua.pt

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FIG. 2. Schematic view of the evolution of an intervertex distance distribution with a growing number of shortcuts between two large networks: (a) two separate networks; (b) two networks with a single shortcut between them; (c) two interconnected networks, when the number of shortcuts is a finite fraction of the number of edges in the networks. $\bar{\ell}_1$ and $\bar{\ell}_2$ are the average intervertex distances in the first and second networks, respectively.

find a distribution of intervertex distances. We show how the global architecture of this (large) network evolves with an increasing number of shortcuts, when two networks merge into one. The question is, when is the mutual equidistance property realized? How general is this feature? Figure 2 schematically presents our result. The conclusion is that the equidistance is realized when the number of shortcuts is a finite fraction of the total number of edges in the network. This finite fraction may be arbitrarily small though bigger than 0. In this respect, the large network becomes united at arbitrarily small concentrations of shortcuts.

In Sec. II we briefly present our results. Section III describes our general approach to these networks based on the Z-transformation (generating function) technique. In Sec. IV we explain how to obtain the intervertex distribution for infinitely large networks. In Sec. V we discuss our results. Finally, in the Appendix we outline the Z-transformation approach in application to the configuration model of uncorrelated networks. In particular, we list useful formulas for the statistics of the nth connected components obtained in Ref. [10].

II. MAIN RESULTS

We apply the theory of Sec. III to the following problem. Two large uncorrelated networks, of N_1 and N_2 vertices, have degree distributions $\Pi_1(q)$ and $\Pi_2(q)$ with converging second moments. (When the second moments diverge, a network has numerous loops, and the tree ansatz becomes dubious. So we do not consider this situation here.) We assume

that dead ends are absent, i.e., $\Pi_1(1)=\Pi_2(1)=0$, which guarantees that finite connected components are not essential in the infinite-network limit (see Refs. [20,21]). $\mathcal{L}$ edges interconnect randomly chosen vertices of net 1 and randomly chosen vertices of net 2. For simplicity, we assume in this problem that $\mathcal{L}$ is much smaller than the total number of connections in the network. The question is, what is the form of the intervertex distance distribution? In the infinite-network limit, to describe this distribution, it is sufficiently to know three numbers: the average distance $\bar{\ell}_1$ between vertices of subnetwork 1, the average distance $\bar{\ell}_2$ between vertices of subnetwork 2, and the average distance $\bar{d}$ between a vertex from subnetwork 1 and a vertex from subnetwork 2. These three numbers give the positions of the three peaks in the distribution. By examining the variations of these three distances with $\mathcal{L}$ one can determine when the equidistance property takes place.

The resulting $\bar{\ell}_1$, $\bar{\ell}_2$, and $\bar{d}$ are given by the following asymptotic $(N_1, N_2 \to \infty)$ formulas:

$$\bar{\ell}_1 \cong \frac{\ln N_1}{\ln \zeta_1}, \tag{1}$$

$$\bar{\ell}_2 \cong \bar{d} + \frac{1}{\ln \zeta_2} \ln[N_2(\zeta_2^{\bar{d}} + C\ell)^{-1}], \tag{2}$$

$$\bar{d} \cong \bar{\ell}_1 + \frac{\ln(N_2/\ell)}{\ln \zeta_2}. \tag{3}$$

Here $\bar{\ell}_1$, $\bar{\ell}_2$, and $\bar{d}$ are expressed in terms of ζ_1 and ζ_2, which generalize the mean branching [ζ in the standard (one-module) configuration model; see Eqs. (A13) in the Appendix]. The expressions (3) are valid if $\zeta_2 < \zeta_1 < \infty$. The constant C is determined by the degree distributions $\Pi_1(q)$ and $\Pi_2(q)$ and is independent of N_1, N_2, and ℓ. In formula (2), $\zeta_2^{\bar{d}} = (N_1 N_2/\ell)^{\ln \zeta_2/\ln \zeta_1}$. Note that these asymptotic estimates ignore constant additives. In their turn, "the mean branchings" ζ_1 and ζ_2 are

$$\zeta_1 = K_1 + \frac{\ell^2}{N_1 N_2 \bar{q}_1 \bar{q}_2} \frac{1}{(K_1 - K_2)K_1^2},$$

$$\zeta_2 = K_2 - \frac{\ell^2}{N_1 N_2 \bar{q}_1 \bar{q}_2} \frac{1}{(K_1 - K_2)K_2^2}, \tag{4}$$

where we assume that $K_1 > K_2$. The quantities K_1 and K_2 are defined as

$$K_{1,2} \equiv \sum_{q,r} q(q-1)\Pi_{1,2}(q,r)/\bar{q}_{1,2}, \tag{5}$$

where $\Pi_{1,2}(q,r)$ are the given distributions of vertices of intradegree q and interdegree r in subnetworks 1 and 2 (see Sec. III for more detail). $\bar{q}_1$ and $\bar{q}_2$ are the mean intradegrees of vertices in subnetworks 1 and 2, respectively.

Formulas (1)–(3) demonstrate that, if $\mathcal{L}$ is a finite fraction of the total number of connections (in the infinite-network limit), then $\bar{\ell}_2$ and $\bar{d}$ approach $\bar{\ell}_1$. The differences are only

finite numbers. Indeed, the second terms in relations (2) and (3) are finite numbers if $N_2/\mathcal{L} \to$ const. [When $\mathcal{L}$ is a finite fraction of the total number of connections, $\zeta_2^{\bar{d}} \ll \mathcal{L}$ in Eq. (2).] On the other hand, when $\mathcal{L}$ is formally set to 1, relation (3) gives $\bar{d} = \bar{\ell}_1 + \bar{\ell}_2$ [see Fig. 2(b)]. Furthermore, assuming $\mathcal{L}/N_{1,2} \to 0$ as $N_{1,2} \to \infty$, we have $\bar{\ell}_2 \cong \ln N_2 / \ln \zeta_2$ according to relation (2), since $\zeta_2^{\bar{d}} = (N_1 N_2/\mathcal{L})^{\ln \zeta_2 / \ln \zeta_1} \gg \mathcal{L}$. Note that the addition of extra links cannot increase any intervertex distance in a network, including the shortest-path distances between vertices within one module. This explains why, in the mutual equidistance regime, the average intervertex distance of the total network collapses approximately to the minimum one [see Fig. 2(c)]. (Recall that here we assume that the number of interlinks is relatively small.)

We also consider a special situation where subnetworks 1 and 2 are equal, so $K_1 = K_2 \equiv K$, $\bar{q}_1 = \bar{q}_2$, and $N_1 = N_2 \equiv N$. In this case the mean branching coefficients are

$$\zeta_{1,2} = K \pm \frac{\ell}{N\bar{q}_1} \frac{1}{K}. \tag{6}$$

With these ζ_1 and ζ_2, the mean intervertex distances have the following asymptotics: $\bar{\ell}_1 = \bar{\ell}_2 \cong \ln N / \ln \zeta_1$ and $\bar{d} \cong \bar{\ell}_1 + \ln(N/\ell)/\ln \zeta_2$. Formally setting $\mathcal{L}$ to 1, we arrive at $\bar{d} = 2\bar{\ell}_1 = 2\bar{\ell}_2$. One should stress that all the listed results indicate a smooth crossover from two separate networks to a single united one: there is no sharp transition between these two regimes.

III. STATISTICS OF MODULAR NETWORKS

We consider two interlinked undirected networks, one of N_1 and the other of N_2 vertices. The adjacency matrix of the joint network, $\hat{g}$, has the structure

$$\hat{g} = \begin{bmatrix} \hat{g}_1 & \hat{h} \\ \hat{h}^T & \hat{g}_2 \end{bmatrix}.$$

Here $\hat{g}_1 = \hat{g}_1^T$ and $\hat{g}_2 = \hat{g}_2^T$ are $N_1 \times N_1$ and $N_2 \times N_2$ adjacency matrices of the first and of the second subnetworks, respectively, and $\hat{h}$ is the $N_1 \times N_2$ matrix for interconnections. We use the following notations: roman (greek) subscripts i, j, etc. (α, β, etc.) take values $1, 2, \ldots, N_1$ ($N_1 + 1, N_1 + 2, \ldots, N_1 + N_2$). So g_{ji}, $g_{\beta\alpha}$, $g_{j\alpha}$, and $g_{\beta i}$ are the matrix elements of $\hat{g}_1$, $\hat{g}_2$, $\hat{h}$, and $\hat{h}^T$, respectively. We assume the whole network to be a simple one, i.e., the matrix elements of $\hat{g}$ are either 0 or 1, and the diagonal ones are all zero, $g_{ii} = g_{\alpha\alpha} = 0$.

Every vertex in this network has an intradegree and interdegree. Vertex i belonging to subnetwork 1 has intradegree $q_i = \Sigma_j g_{ji}$ and interdegree $r_i = \Sigma_\beta g_{\beta i}$. Vertex α belonging to subnetwork 2 has intradegree $q_\alpha = \Sigma_\beta g_{\beta\alpha}$ and interdegree $r_\alpha = \Sigma_j g_{j\alpha}$. The total numbers of intra- and interlinks are $2L_1 = \Sigma_{j,i} g_{ij}$, $2L_2 = \Sigma_{\beta,\alpha} g_{\beta\alpha}$, and $\mathcal{L} = \Sigma_{j,\alpha} g_{j\alpha} = \Sigma_{\beta,i} g_{\beta i}$.

We introduce a natural generalization of the configuration model (we recommend that a reader look over the Appendix to recall the configuration model and the standard analytical approach to the statistics of its components). In our random

network, intralinks in subnetworks 1 and 2 are uncorrelated, and the set of interlinks connecting them is also uncorrelated. As in the configuration model, our statistical ensemble includes all possible networks with given sequences of intra- and interdegrees for both subnetworks. All the members of the ensemble are taken with the same statistical weight. Namely, there are $\mathcal{N}_{1,2}(N_1, N_2; q, r)$ vertices in subnetworks 1,2 of intradegree q and interdegree r. Here $\Sigma_{q,r} \mathcal{N}_{1,2}(N_1, N_2; q, r) = N_{1,2}$. The condition $\Sigma_{q,r} r \mathcal{N}_1(N_1, N_2; q, r) = \Sigma_{q,r} r \mathcal{N}_2(N_1, N_2; q, r) = \mathcal{L}$, where $\mathcal{L}$ is the number of interlinks, should be satisfied. We assume that in the thermodynamic limit $N_1 \to \infty$, $N_2 \to \infty$, $N_2/N_1 \to \kappa < \infty$, we have $\mathcal{N}_{1,2}(N_1, N_2; q, r)/N_{1,2} \to \Pi_{1,2}(q, r)$, where Π_1 and Π_2 are given distribution functions. Again, there is a condition that the number of edges from subnetwork 1 to 2 is the same as from 2 to 1:

$$\bar{r}_1 \equiv \sum_{q,r=0}^{\infty} r\Pi_1(q,r) = \kappa \sum_{q,r=0}^{\infty} r\Pi_2(q,r) \equiv \kappa \bar{r}_2. \tag{7}$$

Here $\bar{r}_{1,2}$ are average interdegrees of the vertices in subnetworks 1 and 2.

The theory of uncorrelated networks extensively uses the Z representation (generating function) of a degree distribution:

$$\phi(x) = \sum_{q=0}^{\infty} \Pi(q) x^q. \tag{8}$$

Here we introduce

$$\phi_{1,2}(x,y) = \sum_{q,r=0}^{\infty} \Pi_{1,2}(q,r) x^q y^r. \tag{9}$$

In Z representation, the average intra- $\bar{q}_1$ and $\bar{q}_2$ and inter- $\bar{r}_1$ and $\bar{r}_2$ degrees of subnetworks 1 and 2, respectively, are

$$\bar{q}_{1,2} = \left. \frac{\partial \phi_{1,2}(x,y)}{\partial x} \right|_{x=y=1}, \quad \bar{r}_{1,2} = \left. \frac{\partial \phi_{1,2}(x,y)}{\partial y} \right|_{x=y=1}. \tag{10}$$

In the unimodal configuration model, the statistics of components was obtained based on the distribution of branching of an edge, $\tilde{\Pi}(q)$ [see Eq. (A1)]. This is the probability that an end vertex of a randomly chosen edge has degree $q - 1$. In contrast to this, there are the four following possibilities for the branching of an edge in a network with two modules. (i) An edge is in subnetwork 1; (ii) an edge is in subnetwork 2; (iii) an edge connects the modules, and we consider the branching at the end belonging to subnetwork 1; (iv) for this edge, we consider the branching at the end belonging to subnetwork 2. In addition, in each of these four cases, we must distinguish the intra- and interbranching. These are the numbers of intra- and interconnections emanating from a given end of a given edge, respectively. Consequently, we should introduce four different distributions of intra- and interbranching.

Let (j, i), (β, α), (j, α), and (β, i) be ordered vertex pairs. Let us name their elements in the first and second positions as final and initial, respectively. The joint intra- and inter-

branching distribution is the conditional probability for the final vertex of some (randomly chosen) ordered pair of vertices to have intra- and interbranchings q and r, respectively, provided the vertices in this pair are connected by an edge. To be clear, q and r are the numbers of the intra- and interconnections of the final vertex, excluding the edge connecting the pair of vertices. So we have four distributions, each one depending on two variables:

$$P_1(q,r) = \frac{1}{2L_1} \sum_{ji} \langle g_{ji} \delta(q_j - 1 - q) \delta(r_j - r) \rangle,$$

$$P_2(q,r) = \frac{1}{2L_2} \sum_{\beta,\alpha} \langle g_{\beta\alpha} \delta(q_\beta - 1 - q) \delta(r_\beta - r) \rangle,$$

$$Q_1(q,r) = \frac{1}{\mathcal{L}} \sum_{j,\alpha} \langle g_{j\alpha} \delta(q_j - q) \delta(r_j - 1 - r) \rangle,$$

$$Q_2(q,r) = \frac{1}{\mathcal{L}} \sum_{\beta,i} \langle g_{\beta i} \delta(q_\beta - q) \delta(r_\beta - 1 - r) \rangle. \quad (11)$$

Note that some of the δ functions on the right-hand sides of these definitions have -1 in their arguments and others have not. In other words, in some cases the intra- or interbranching is smaller than the intra- or interdegree of the end vertex by 1. In other cases, the intra- or interdegree branching coincides with the intra- or interdegree, respectively. Let us explain this difference, e.g., for $P_1(q,r)$. In this case, the edge is in module 1. So the intrabranching at any of its ends equals the intradegree of this end vertex minus 1, since the edge itself must be subtracted. In contrast, the interbranching coincides with the interdegree of the relevant end vertex. Indeed, in this case, the edge does not contribute to the interdegree. Similar arguments explain the definitions of the other three distributions. Taking into account the definitions of vertex degrees, we have

$$P_{1,2}(q,r) = (q+1)\Pi_{1,2}(q+1,r)/\bar{q}_{1,2},$$

$$Q_{1,2}(q,r) = (r+1)\Pi_{1,2}(q,r+1)/\bar{r}_{1,2}. \quad (12)$$

In Z representation these distribution functions take the following forms:

$$\xi_1(x,y) = \frac{1}{2L_1} \sum_j \langle q_j x^{q_j-1} y^{r_j} \rangle = \frac{1}{\bar{q}_1} \frac{\partial \phi_1(x,y)}{\partial x},$$

$$\xi_2(x,y) = \frac{1}{2L_2} \sum_\beta \langle q_\beta x^{q_\beta-1} y^{r_\beta} \rangle = \frac{1}{\bar{q}_2} \frac{\partial \phi_2(x,y)}{\partial x},$$

$$\eta_1(x,y) = \frac{1}{\mathcal{L}} \sum_j \langle r_j x^{q_j} y^{r_j-1} \rangle = \frac{1}{\bar{r}_1} \frac{\partial \phi_1(x,y)}{\partial y},$$

$$\eta_2(x,y) = \frac{1}{\mathcal{L}} \sum_\beta \langle r_\beta x^{q_\beta} y^{r_\beta-1} \rangle = \frac{1}{\bar{r}_2} \frac{\partial \phi_2(x,y)}{\partial y}. \quad (13)$$

Let us introduce the nth components of ordered vertex pairs $C_{n,ji}$, $C_{n,\beta\alpha}$, $C_{n,j\alpha}$, and $C_{n,\beta i}$. These components are sets,

whose elements are vertices. As is natural, the components are empty, if the vertices in a pair are not connected. The first component is either a one-element set consisting of the final vertex, or an empty set. For example, $C_{1,\beta i}$ is either vertex β or $\varnothing$. The second component, if nonempty, contains also all the nearest neighbors of the final vertex, except the initial one, and so on. We have four types of the components of an edge: $C_{n,ji}$, $C_{n,\beta\alpha}$, $C_{n,j\alpha}$, and $C_{n,\beta i}$. They are defined in a recursive way similarly to the standard configuration model (see the Appendix). Each of these four nth components itself consists of two disjoint sets: one of the vertices in subnetwork 1, the other in subnetwork 2. For example, $C_{n,ji} = C_{n,ji}^{(1)} \cup C_{n,ji}^{(2)}$.

The sizes of the components are $M_{n,ji}^{(1)} = |C_{n,ji}^{(1)}|$, etc. Taking into account the locally treelike structure of our network gives

$$M_{n,ji}^{(1,2)} = g_{ji} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{k \neq i} M_{n-1,kj}^{(1,2)} + \sum_\gamma M_{n-1,\gamma j}^{(1,2)} \right],$$

$$M_{n,\beta\alpha}^{(1,2)} = g_{\beta a} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_k M_{n-1,k\beta}^{(1,2)} + \sum_{\gamma \neq \alpha} M_{n-1,\gamma\beta}^{(1,2)} \right],$$

$$M_{n,j\alpha}^{(1,2)} = g_{ja} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_k M_{n-1,kj}^{(1,2)} + \sum_{\gamma \neq \alpha} M_{n-1,\gamma j}^{(1,2)} x \right],$$

$$M_{n,\beta i}^{(1,2)} = g_{\beta i} \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{k \neq i} M_{n-1,k\beta}^{(1,2)} + \sum_\gamma M_{n-1,\gamma\beta}^{(1,2)} \right]. \quad (14)$$

The configuration model is an uncorrelated random network. So all the terms on the right-hand side of each of the four equations (14) are independent random variables. Quantities within each of two sums in these equations are equally distributed. Their statistical properties are also independent of the degree distribution of the initial vertex of the edge, i.e., of j or β.

The sizes of the connected components of an edge in different networks (e.g., $M_{n,jk}^{(1)}$ and $M_{n,jk}^{(2)}$) are, generally, correlated. So we introduce four joint distribution functions of the component sizes in different networks. In Z representation, they are defined as follows:

$$\psi_n^{(1)}(x,y) = \frac{1}{2L_1} \left\langle \sum_{j,i} g_{ji} x^{M_{n,ji}^{(1)}} y^{M_{n,ji}^{(2)}} \right\rangle,$$

$$\psi_n^{(2)}(x,y) = \frac{1}{2L_2} \left\langle \sum_{\beta,\alpha} g_{\beta\alpha} x^{M_{n,\beta\alpha}^{(1)}} y^{M_{n,\beta\alpha}^{(2)}} \right\rangle,$$

$$\theta_n^{(1)}(x,y) = \frac{1}{\mathcal{L}} \left\langle \sum_{j,\alpha} g_{j\alpha} x^{M_{n,j\alpha}^{(1)}} y^{M_{n,j\alpha}^{(2)}} \right\rangle,$$

$$\theta_n^{(2)}(x,y) = \frac{1}{\mathcal{L}} \left\langle \sum_{\beta,i} g_{\beta i} x^{M_{n,\beta i}^{(1)}} y^{M_{n,\beta i}^{(2)}} \right\rangle. \quad (15)$$

The recursive relations for these distributions are straightforward generalizations of a relation for the usual uncorrelated

network, without modularity [see Eq. (A7) in the Appendix],

$$\psi_n^{(1)}(x,y) = x\xi_1[\psi_{n-1}^{(1)}(x,y), \theta_{n-1}^{(2)}(x,y)],$$

$$\psi_n^{(2)}(x,y) = y\xi_2[\psi_{n-1}^{(2)}(x,y), \theta_{n-1}^{(1)}(x,y)],$$

$$\theta_n^{(1)}(x,y) = x\eta_1[\psi_{n-1}^{(1)}(x,y), \theta_{n-1}^{(2)}(x,y)],$$

$$\theta_n^{(2)}(x,y) = y\eta_2[\psi_{n-1}^{(2)}(x,y), \theta_{n-1}^{(1)}(x,y)]. \quad (16)$$

The nth component $\mathcal{C}_{n,i}$ ($\mathcal{C}_{n,\alpha}$) of vertex i (α) contains all vertices at distance n from vertex i (α) or closer. Let $\mathcal{M}_{n,i}^{(1,2)} = |\mathcal{C}_{n,i}^{(1,2)}|$ and $\mathcal{M}_{n,\alpha}^{(1,2)} = |\mathcal{C}_{n,\alpha}^{(1,2)}|$ be the sizes of the components ($\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are the subsets of $\mathcal{C}$, containing vertices of the first and second networks, respectively). Using the locally treelike structure of the network and absence of correlations between its vertices, we obtain the Z transform of the joint distributions of component sizes:

$$\Psi_n^{(1)}(x,y) = \frac{1}{N_1}\left\langle \sum_i x^{\mathcal{M}_{n,i}^{(1)}} y^{\mathcal{M}_{n,i}^{(2)}} \right\rangle = x\phi_1[\psi_n^{(1)}(x,y), \theta_n^{(1)}(x,y)],$$

$$\Psi_n^{(2)}(x,y) = \frac{1}{N_2}\left\langle \sum_\alpha x^{\mathcal{M}_{n,\alpha}^{(1)}} y^{\mathcal{M}_{n,\alpha}^{(2)}} \right\rangle = y\phi_2[\psi_n^{(2)}(x,y), \theta_n^{(2)}(x,y)].$$

$$(17)$$

The conditional average sizes of the components are expressed in terms of the derivatives of the corresponding distribution functions at the point $x=y=1$. For example, the conditional average component sizes for an internal vertex pair in network 1 are expressed as follows: for the part, which belongs to the first network it is

$$\bar{M}_n^{(111)} = \langle g_{ij}M_{n,ij}^{(1)}\rangle/\langle g_{ij}\rangle = \left.\frac{\partial\psi_n^{(1)}(x,y)}{\partial x}\right|_{x=y=1} \quad (18)$$

for the component part in network 1, and

$$\bar{M}_n^{(211)} = \langle g_{ij}M_{n,ij}^{(2)}\rangle/\langle g_{ij}\rangle = \left.\frac{\partial\psi_n^{(1)}(x,y)}{\partial y}\right|_{x=y=1} \quad (19)$$

for the part of the component which is in network 2. Here, (i) the first superscript indicates whether the component is in subnetwork 1 or 2, (ii) the second superscript indicates whether the final vertex is in subnetwork 1 or 2, and (iii) the third superscript indicates whether the initial vertex is in subnetwork 1 or 2. For the components of a pair with initial vertex in network 2 and final in network 1 we have

$$\bar{M}_n^{(112)} = \langle g_{i\alpha}M_{n,i\alpha}^{(1)}\rangle/\langle g_{i\alpha}\rangle = \left.\frac{\partial\theta_n^{(1)}(x,y)}{\partial x}\right|_{x=y=1} \quad (20)$$

and

$$\bar{M}_n^{(212)} = \langle g_{i\alpha}M_{n,i\alpha}^{(2)}\rangle/\langle g_{i\alpha}\rangle = \left.\frac{\partial\theta_n^{(1)}(x,y)}{\partial y}\right|_{x=y=1}, \quad (21)$$

and so on.

Using Eqs. (16) one can derive recurrent relations for the average values of M_n and M_{n-1}. We introduce a pair of four-dimensional vectors:

$$\mathbf{M}_n^{(1)} = \begin{pmatrix} \bar{M}_n^{(111)} \\ \bar{M}_n^{(112)} \\ \bar{M}_n^{(121)} \\ \bar{M}_n^{(122)} \end{pmatrix}, \quad \mathbf{M}_n^{(2)} = \begin{pmatrix} \bar{M}_n^{(211)} \\ \bar{M}_n^{(212)} \\ \bar{M}_n^{(221)} \\ \bar{M}_n^{(222)} \end{pmatrix}. \quad (22)$$

Then the recurrent relations take the forms

$$\mathbf{M}_n^{(1)} = \hat{\zeta}\mathbf{M}_{n-1}^{(1)} + \mathbf{m}_1, \quad \mathbf{M}_n^{(2)} = \hat{\zeta}\mathbf{M}_{n-1}^{(2)} + \mathbf{m}_2, \quad (23)$$

where

$$\hat{\zeta} = \begin{pmatrix} \xi_{11} & 0 & \xi_{21} & 0 \\ \eta_{11} & 0 & \eta_{21} & 0 \\ 0 & \eta_{22} & 0 & \eta_{12} \\ 0 & \xi_{22} & 0 & \xi_{12} \end{pmatrix}, \quad \mathbf{m}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{m}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

$$(24)$$

Here

$$\xi_{\mu\nu} = \left.\partial_\mu\xi_\nu(x,y)\right|_{x=y=1}, \quad \eta_{\mu\nu} = \left.\partial_\mu\eta_\nu(x,y)\right|_{x=y=1},$$

where, $\mu,\nu=1,2$. The initial conditions are $\mathbf{M}_1^{(1)}=\mathbf{m}_1$, $\mathbf{M}_1^{(2)}=\mathbf{m}_2$. As for the average sizes of the nth components of vertices, they are

$$\bar{\mathcal{M}}_n^{(11)} = \frac{1}{N_1}\left\langle \sum_i \mathcal{M}_{n,i}^{(1)} \right\rangle = \left.\frac{\partial\Psi_n^{(1)}(x,y)}{\partial x}\right|_{x=y=1} = 1 + \bar{q}_1\bar{M}_n^{(111)}$$
$$+ \bar{r}_1\bar{M}_n^{(112)},$$

$$\bar{\mathcal{M}}_n^{(21)} = \frac{1}{N_1}\left\langle \sum_i \mathcal{M}_{n,i}^{(2)} \right\rangle = \left.\frac{\partial\Psi_n^{(1)}(x,y)}{\partial y}\right|_{x=y=1} = \bar{q}_1\bar{M}_n^{(211)}$$
$$+ \bar{r}_1\bar{M}_n^{(212)},$$

$$\bar{\mathcal{M}}_n^{(12)} = \frac{1}{N_1}\left\langle \sum_\alpha \mathcal{M}_{n,i}^{(1)} \right\rangle = \left.\frac{\partial\Psi_n^{(2)}(x,y)}{\partial x}\right|_{x=y=1} = \bar{q}_2\bar{M}_n^{(122)}$$
$$+ \bar{r}_2\bar{M}_n^{(121)},$$

$$\bar{\mathcal{M}}_n^{(22)} = \frac{1}{N_2}\left\langle \sum_i \mathcal{M}_{n,i}^{(2)} \right\rangle = \left.\frac{\partial\Psi_n^{(2)}(x,y)}{\partial y}\right|_{x=y=1} = 1 + \bar{q}_2\bar{M}_n^{(222)}$$
$$+ \bar{r}_2\bar{M}_n^{(221)}. \quad (25)$$

Here, (i) the first superscript of $\bar{\mathcal{M}}_n$ indicates whether the component is in subnetwork 1 or 2, and (ii) the second superscript indicates whether a mother vertex is in subnetwork 1 or 2. Recall that $\bar{q}_1$ and $\bar{q}_2$ are the mean numbers of internal connections of vertices in subnetworks 1 and 2, respectively; $\bar{r}_1$ is the mean number of connections of a vertex in subnetwork 1, which go to subnetwork 2; and finally $\bar{r}_2$ is the mean number of connections of a vertex in subnetwork 2, which go to subnetwork 1. Relations (23) and (25) allow us to obtain the average sizes of all components.

Example

Since formulas in this section are rather cumbersome, to help the readers, we present a simple demonstrative example of the application of these relations. Let us describe the emergence of a giant connected component in a symmetric situation, where both subnetworks have equal sizes and identical degree distributions $\Pi_{1,2}(q,r) \equiv \Pi(q,r)$. In this case, $\xi_1(x,y) = \xi_2(x,y) \equiv \xi(x,y)$ and $\eta_1(x,y) = \eta_2(x,y) \equiv \eta(x,y)$. Also, $\psi^{(1)}(x,y) = \psi^{(2)}(x,y) \equiv \psi(x,y)$ and $\theta^{(1)}(x,y) = \theta^{(2)}(x,y) \equiv \theta(x,y)$. So the relative size S of a giant connected component takes the form

$$S = 1 - \phi(t,u), \qquad (26)$$

where $t \equiv \psi(1,1)$ and $u \equiv \theta(1,1)$ are nontrivial solutions of the equations:

$$t = \xi(t,u), \quad u = \eta(t,u). \qquad (27)$$

For example, let the subnetworks be classical random graphs, and each vertex has no interlinks with a probability $1-p$ and has a single interlink with the complementary probability p. That is,

$$\Pi(q,r) = e^{-\bar{q}} \frac{\bar{q}^q}{q!} [(1-p)\delta_{r,0} + p\delta_{r,1}], \qquad (28)$$

where $\bar{q}$ is the mean vertex intradegree, so $\phi(x,y) = e^{\bar{q}(x-1)}(1-p+py)$.

For a single classical random graph with vertices of average degree $\bar{q}$, Eqs. (A15) and (A17) give the point of emergence of a giant connected component, $\bar{q} = q_c = 1$, and the relative size of this component $S \cong 2(\bar{q}-1)$ in the critical region.

Let us now find the emergence point $\bar{q} = q_c(p)$ and the critical dependence $S(\bar{q},p)$ in the modular network. For this network, we find $\xi(x,y) = \partial_x \phi(x,y)/\bar{q} = \phi(x,y)$ and $\eta(x,y) = \partial_y \phi(x,y)/\bar{r} = e^{\bar{q}(x-1)}$. Substitution of these functions into Eqs. (26) and (27) directly leads to the result

$$q_c = \frac{1}{1+p}, \quad S \cong 2\frac{(1+p)^3}{1+3p}\left(\bar{q} - \frac{1}{1+p}\right), \qquad (29)$$

compare with a single classical random graph.

IV. INTERVERTEX DISTANCE DISTRIBUTION

As was explained in Sec. II, the intervertex distance distribution in the thermodynamic limit is completely determined by the three mean intervertex distances $\bar{\ell}_1$ for subnetwork 1, $\bar{\ell}_2$ for subnetwork 2, and $\bar{d}$ for pairs of vertices where the first vertex is in subnetwork 1 and the second is in subnetwork 2. The idea of the computation of these intervertex distances is very similar to that in the standard configuration model [see the Appendix, Eq. (18)]. However, the straightforward calculations for two interconnected networks are cumbersome, so here we only indicate some points in our derivations without going into technical details.

The calculations are based on the solution of the recursive relations (23). As is usual, these relations should be investigated in the range $1-x \ll 1$, $1-y \ll 1$ of the Z-transformation

parameters. Fortunately, the problem can be essentially reduced to the calculation of the two highest eigenvalues of a single 4×4 matrix. The resulting eigenvalues ζ_1 and ζ_2 for networks with $\ell/N_{1,2} \ll 1$ are given by formulas (4) and (6). The nth component sizes are expressed in terms of these eigenvalues. The leading contributions to the nth component sizes turn out to be linear combinations of powers of the mean branchings: $A\zeta_1^n + B\zeta_2^n$. The factors A and B do not depend on n. For example, when $\zeta_1 > \zeta_2$, the main contributions to $\bar{\mathcal{M}}_n^{(11)}$ and $\bar{\mathcal{M}}_n^{(21)}$ are $\zeta_1^n + [\ell^2/(N_1 N_2)]\zeta_2^n$ and $(\mathcal{L}/N_1)\zeta_2^n$, respectively. Here we omitted nonessential factors and assumed a large n. This approximation is based on the tree ansatz, that is, on the locally treelike structure of the network. This ansatz works when the nth components are much smaller than subnetworks 1 and 2. So the intervertex distances are obtained by comparing the sizes of relevant nth components of vertices with N_1 and N_2. Since networks 1 and 2 are uncorrelated, this estimate gives only a constant additive error which is much smaller than the main contribution of the order of $\ln N_{1,2}$. (See Ref. [10] for complicated calculations beyond the tree ansatz in the standard configuration model, which allow one to obtain this constant number.)

One should emphasize an additional difficulty specific for the networks under consideration. The problem is that in some range of N_1 and N_2, and ζ_1 and ζ_2, while an nth component in, say, network 1 is already of size $\sim N_1$ (failing tree ansatz), the corresponding nth component in network 2 is still much smaller than N_2. In terms of Sec. III, this, e.g., means that there exists a range of n, $\bar{\ell}_1 < n < \bar{d}$, where $\bar{\mathcal{M}}_n^{(11)} \sim N_1$ but still $\bar{\mathcal{M}}_n^{(21)} \ll N_2$. Computing $\bar{\mathcal{M}}_n^{(21)}$ in this regime, we use the tree ansatz, while $\bar{\mathcal{M}}_n^{(11)}$ is set to N_1. This approximation also produces only a constant additive error which one may ignore in these asymptotic estimates.

V. DISCUSSION AND CONCLUSIONS

A few points should be stressed.

(i) In Sec. III we derived relations for the Z transformation of the distributions of nth components. Quite similarly to the standard configuration model (see the Appendix), using these formulas with $n \to \infty$ readily gives corresponding relations for the statistics of finite connected components and for the size of a giant connected component. Note that when subnetworks 1 and 2 are uncorrelated, which is our case, finite connected components are essential in an infinite network only if there is a finite fraction of vertices of degree 1.

(ii) The theory of Sec. III is essentially based on the locally treelike structure of networks under consideration. In principle one can go even beyond the tree ansatz as was done for the standard configuration model in Ref. [10]. This is a challenging problem for these networks. Since we extensively used the tree approximation in Sec. IV, our results for the intervertex distances are only asymptotic estimates.

(iii) For the sake of brevity, we obtained relations only for networks with two interlinked subnetworks, but it is not a restriction. A generalization to networks with an arbitrary number of interlinked subnetworks is straightforward. The

final relations in Sec. III can be readily generalized without derivation. Generalization to structured networks with degree-degree correlations is also clear. Note that, in particular, our theory can describe multi-partite networks, whose subnetworks have no intraconnections. Based on equations derived for the configuration model [22,23] (for k cores in real-world networks; see Refs. [24,25]), one can also generalize this theory to describe the k-core organization of modular networks

(iv) As an application, we considered interlinked networks with a relatively small number of shortcuts. Note, however, that our general results in Sec. III do not assume this restriction.

In summary, we have developed an analytical approach to the statistics of networks with well-distinguished communities. We have derived general relations allowing one to find the distributions of the sizes of connected components in these networks. As a particular application of this theory, we have obtained asymptotic estimates for the distribution of intervertex distances in two weakly interconnected uncorrelated networks. We have shown that, in the infinite-network limit, vertices in this network are almost surely equidistant if the relative number of interlinks is any finite number. Our approach can be applied to a number of other problems for networks of this sort, including the emergence of a giant connected component, percolation, and others.

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APPENDIX: STATISTICS OF THE CONFIGURATION MODEL

For the sake of clarity, here we outline the Z-transformation (generating function) technique in application to the standard configuration model of an uncorrelated graph with a given degree distribution $\Pi(q)$. For more detail, see Refs. [8–10]. Note that we use this technique to derive the nth connected component statistics [10] unlike earlier approaches [8,9] where only the size distribution of connected components was obtained. In Sec. III we generalize the description of the statistics of the nth connected component to modular networks. In simple terms, the configuration model [18,19] is a maximally random graph with a given degree distribution. In graph theory it is also called a random graph with a given degree sequence.

A graph of size N consists of a set of vertices v_i, $i = 1, 2, \ldots, N$, connected by edges e_{ji}. An edge e_{ji} exists if the adjacency matrix element $g_{ji} = 1$. We start from the following distribution:

$$\tilde{\Pi}(q) = \frac{1}{2L} \left\langle \sum_{ji=1}^{N} g_{ji} \delta_K(q_j - 1 - q) \right\rangle, \quad \text{(A1)}$$

where δ_K is the Kronecker symbol. This is the probability that a randomly chosen end of a randomly chosen edge in the
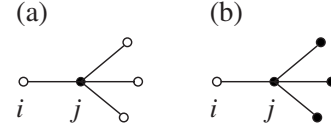

FIG. 3. First (a) and second (b) components of edge (ij). Filled vertices belong to the components. The zeroth component is empty.

graph has branching q (degree of the end vertex minus 1). Alternatively, it may be considered as the conditional probability for the final vertex j in a randomly chosen ordered pair (j, i) to have degree $q+1$, provided the vertices are connected by an edge. Obviously,

$$\tilde{\Pi}(q) = \frac{1}{2L} \left\langle \sum_{j=1}^{N} q_j \delta_K(q_j - 1 - q) \right\rangle = \frac{q+1}{\bar{q}} \Pi(q+1), \quad \text{(A2)}$$

where $\bar{q} \equiv \langle q \rangle$ is the average degree of a vertex. In the Z representation this relation takes the form

$$\tilde{\phi}(x) = \sum_{q=0}^{\infty} \tilde{\Pi}(q) x^q = \frac{1}{2L} \left\langle \sum_{j=1}^{N} q_j x^{q_j - 1} \right\rangle = \frac{\phi'(x)}{\bar{q}}. \quad \text{(A3)}$$

Note that $\bar{q} = \phi'(1)$.

Let the nth component of the ordered pair (j, i), $C_{n,ji}$, be the following set of vertices. For any (ordered) pair of vertices (j, i), $C_{1,ji}$ is vertex j if the vertices are connected, else $C_{1,ji} = \varnothing$. For $n > 1$, $C_{n,ij}$ is defined recursively as follows. If $g_{ji} = 0$, all $C_{n,ji} = \varnothing$. Otherwise, in $C_{2,ij}$ there are also $q_j - 1$ other vertices, connected to the vertex j, the third component $C_{3,ij}$ contains also all other vertices, connected with ones of the second component, and so on (see Fig. 3).

In the thermodynamic limit ($N \to \infty$) almost every finite nth component of an uncorrelated random graph is a tree. Then for the sizes (numbers of vertices) of the components, $M_{n,ij} = |C_{n,ij}|$, we have (assuming $g_{ji} = 1$)

$$M_{n,ji} = 1 + \sum_{k \neq i} M_{n-1,kj}, \quad \text{(A4)}$$

with the initial condition $M_{1,ji} = 1$. Due to the absence of correlations in the configuration model, $M_{n,kj}$ and $M_{n,lj}$, $k \neq l$, are independent equally distributed random variables. We define the distribution function of the nth component of an edge as

$$p_n(M) = \frac{1}{2L} \sum_{j,i=1}^{N} \langle g_{ji} \delta(M_{n,ij} - M) \rangle$$

$$= \frac{N(N-1)}{2L} \langle g_{ji} \delta(M_{n,ji} - M) \rangle$$

$$= \frac{1}{\langle g_{ji} \rangle} \langle g_{ji} \delta(M_{n,ji} - M) \rangle. \quad \text{(A5)}$$

It is more convenient to use the Z transformation of this distribution:

$$\psi_n(x) = \frac{1}{2L}\left\langle \sum_{i,j=1}^{N} g_{ij} x^{M_{n,ij}} \right\rangle = \frac{1}{\langle g_{ij}\rangle}\langle g_{ij} x^{M_{n,ij}}\rangle. \quad \text{(A6)}$$

Substituting Eq. (A4) into Eq. (A6) and using Eq. (A3), we obtain

$$\psi_n(x) = \frac{x}{2L}\left\langle \sum_{j,i=1}^{N} g_{ji} \prod_{k\neq i} \frac{1}{\langle g_{kj}\rangle}\langle g_{kj} x^{M_{n-1,kj}}\rangle \right\rangle$$

$$= \frac{x}{2L}\left\langle \sum_{j=1}^{N} q_j [\psi_{n-1}(x)]^{q_j-1} \right\rangle = x\tilde{\phi}[\psi_{n-1}(x)]. \quad \text{(A7)}$$

Let $\mathcal{C}_{n,i}$ be the nth component of vertex v_i. This component includes all vertices at distance n or closer from vertex v_i. (The zeroth component of a vertex is empty.) Due to the absence of loops (treelike structure), we have the following relation for the size of the nth component of vertex v_i, $\mathcal{M}_{n,i} = |\mathcal{C}_{n,i}|$;

$$\mathcal{M}_{n,i} = 1 + \sum_j M_{n-1,ij}. \quad \text{(A8)}$$

So the nth component size distribution

$$P_n(\mathcal{M}) = \frac{1}{N}\left\langle \sum_{i=1}^{N} \delta(\mathcal{M}_{n,i} - \mathcal{M}) \right\rangle \quad \text{(A9)}$$

is expressed in Z representation as

$$\Psi_n(x) = \frac{1}{N}\left\langle \sum_{i=1}^{N} x^{\mathcal{M}_{n,i}} \right\rangle = x\phi[\psi_{n-1}(x)]. \quad \text{(A10)}$$

The average sizes of subsequent nth components are related through the following equations:

$$\bar{\mathcal{M}}_n = \Psi_n'(1) = 1 + \bar{q}\bar{M}_{n-1}, \quad \text{(A11)}$$

$$\bar{M}_n = 1 + \zeta\bar{M}_{n-1}, \quad \text{(A12)}$$

where

$$\zeta \equiv \tilde{\phi}(1) = \frac{\phi''(1)}{\bar{q}} = \frac{1}{\bar{q}}\sum_{q=0}^{\infty} q(q-1)\Pi(q) = \frac{\langle q^2\rangle}{\bar{q}} - 1,$$
$$\text{(A13)}$$

which is the mean branching. If $\zeta < 1$, both M_n and $\mathcal{M}_n$ have finite limits as $n \to \infty$. That is, the network has no giant connected component. If $\zeta > 1$, a giant connected component exists.

Assuming $\psi_n = \psi_{n-1} \equiv \psi$ in Eq. (A7), we obtain an equation for the distribution function of the sizes of edge's connected components,

$$\psi(x) = x\tilde{\phi}[\psi(x)], \quad \text{(A14)}$$

which implicitly defines $\psi(x)$. If $\zeta > 1$, this equation has two solutions at $x=1$. One is $\psi(1)=1$, the other is some $\psi(1) \equiv t < 1$,

$$t = \tilde{\phi}(t). \quad \text{(A15)}$$

For any value of ζ, $\psi_n(1)=1$. On the other hand, if $\zeta > 1$, $\lim_{x\to1-0}\lim_{n\to\infty}\psi_n(x) = t < 1$. This is the probability that the connected component of a randomly chosen edge is finite. Then the probability that a randomly chosen vertex belongs to a finite connected component of the graph is

$$\sum_{q=0}^{\infty} \Pi(q)t^q = \phi(t). \quad \text{(A16)}$$

Therefore the number of vertices in the giant connected component in the thermodynamic limit is

$$M_\infty = N[1 - \phi(t)]. \quad \text{(A17)}$$

One may find the intervertex distance distribution from the mean sizes of the nth components of a vertex (see Refs. [8,26]). The diameter $\bar{\ell}$ of the giant connected component, i.e., the distance between two randomly chosen vertices, is obtained from the relation $\zeta \sim M_\infty \sim N$. So, if the second moment of the degree distribution is finite,

$$\bar{\ell} \cong \frac{\ln aN}{\ln \zeta}, \quad \text{(A18)}$$

where a is some number of the order of 1. For more straightforward calculations, see Refs. [10,27].

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infinite small worlds in the following way. The mean number of ℓth nearest neighbors of a vertex in small worlds usually grows with ℓ as ζ, where ζ is the mean branching coefficient, and $\ell \lesssim \bar{\ell}(N)$. This leads to the following estimate for the intervertex distance distribution: $\mathcal{P}(\ell,N) \propto \zeta\theta(\bar{\ell}(N)-\ell)$. In addition, one must take into account normalization, which readily gives $\mathcal{P}(\ell,N) \to \delta(\ell-\bar{\ell}(N))$ as $N \to \infty$.

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